

Fractional Almost Kähler – Lagrange Geometry

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Abstract

The goal of this paper is to encode equivalently the fractional Lagrange dynamics as a nonholonomic almost Kähler geometry. We use the fractional Caputo derivative generalized for nontrivial nonlinear connections (N-connections) originally introduced in Finsler geometry, with further developments in Lagrange and Hamilton geometry. For fundamental geometric objects induced canonically by regular Lagrange functions, we construct compatible almost symplectic forms and linear connections completely determined by a "prime" Lagrange (in particular, Finsler) generating function. We emphasize the importance of such constructions for deformation quantization of fractional Lagrange geometries and applications in modern physics.

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1 Introduction

The fractional calculus and geometric mechanics are in areas of many important researches in modern mathematics and applications. During the

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last years, various developments of early approaches to fractional calculus [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] were reported in various branches of science and engineering. The fractional calculus is gaining importance due to its superior results in describing the dynamics of complex systems [6, 12, 8]. One important branch of fractional calculus is devoted to the fractional variational principles and their applications in physics and control theory [20, 21, 22, 23, 24]. One of the open problems in the area of fractional calculus is to find an appropriate geometrization of the fractional Lagrange mechanics.

In our paper we provide a geometrization of the fractional Lagrange mechanics using methods of Lagrange–Finsler and (almost) Kähler geometry following the ideas and constructions from two partner works [28, 29] which initiated a new direction of fractional differential geometries and related mathematical relativity/gravity theory and nonholonomic Ricci flow evolution. Quantization of such theories is an interesting and challenging problem connected to fundamental issues in modern quantum gravity and fundamental interaction/ evolution theories.

There were published different papers related to quantum fractional theories (see, for instance, Refs. [25, 26, 27]). Nevertheless, there is not a general/systematic approach to quantization of fractional field/evolution theories. The aim of our third partner work [30] is to prove that the strategy used in Refs. [31, 32, 33, 34] allows us to quantize also a class of fractional geometries following Fedosov method [35, 36]. To perform such a program is necessary to elaborate (in this paper) a general fractional version of almost Kähler geometry and then to apply (in the next paper [30]) the Karabegov–Schlichenmaier deformation quantization scheme [37]. For the canonical models (with the so-called canonical Cartan–Finsler distinguished connections [38]) of Finsler and Lagrange geometries [39], there is a standard scheme of encoding such spaces into almost Kähler geometries (the result is due to Matsumoto [40], for Finsler geometry, and then extended to Lagrange spaces [41]; see further developments in [43, 42]).

In brief, a geometrization of a physical theory in terms of an almost Kähler geometry means a "natural" encoding of fundamental physical objects into terms of basic geometric objects of an almost symplectic geometry (with certain canonical one- and two-forms and connections, almost complex structure etc induced in a unique form, for instance, by a Lagrange or Hamilton function). In general, physical theories are with nonholonomic (equivalently, anholonomic, or non-integrable) structures which does not allow us to transform realistic physical models into (complex) symplect/Kähler geometries. Nevertheless, the bulk of physical theories can be modelled

as almost symplectic geometries for which there were elaborated advanced mathematical approaches and formalism of geometric/deformation quantization, spinor and noncommutative generalizations, unification schemes etc. Our main idea is to show that certain versions of fractional calculus (based on Caputo derivative), and related physical models, admit almost Kähler type fractional geometric encoding which provides an unification scheme with integer and non-integer derivatives for classical and quantum physical theories.

In our papers [28, 29, 30], we work with the Caputo fractional derivative. The advantage of using this fractional derivative (resulting in zero values for actions on constants) is that it allows us to develop a fractional differential geometry which is quite similar to standard (integer dimensions) models and to use standard boundary conditions of the calculus of variations. There is also a series expansion formalism for the fractional calculus and fractional differential equations [44] allowing us to compute fractional modifications of formulas in quite simple forms. This explains why such constructions are popular in engineering and physics.¹

The plan of this paper is as follows: In section 2, we outline the necessary formulas on fractional Caputo calculus and related geometry of tangent bundles. A brief formulation of fractional Lagrange–Finsler geometry is presented in section 3. The main results on almost Kähler models of fractional Lagrange spaces are provided in section 4. Finally, conclusions are presented in section 5.

2 Preliminaries: Fractional Caputo Calculus on Tangent Bundles

We follow the conventions established in Refs. [28, 29].² In our approach, we try to elaborate fractional geometric models with nonholonomic distributions when the constructions are most closed to "integer" calculus.

Let $f(x)$ be a derivable function $f : [{}_1x, {}_2x] \rightarrow \mathbb{R}$, for $\mathbb{R} \ni \alpha > 0$, and $\partial_x = \partial/\partial x$ be the usual partial derivative. By definition, the left Riemann–

¹Perhaps, it is not possible to introduce a generally accepted definition of fractional derivative. For constructions with various types of distributions, in different theories of physics, economics, biology etc, when the geometric picture is less important, it may be more convenient to use the Riemann–Liouville (RL) fractional derivative.

²All left, "up" and/or "low" indices will be considered as labels for some geometric objects but the right ones as typical abstract or coordinate indices which may rung values for corresponding space/time dimensions. Boldface letters will be used for spaces endowed with nonlinear connection structure.

Liouville (RL) derivative is

$${}_1x\overset{\alpha}{\partial}_x f(x) := \frac{1}{\Gamma(s-\alpha)} \left(\frac{\partial}{\partial x} \right)^s \int_{1x}^x (x-x')^{s-\alpha-1} f(x') dx',$$

where Γ is the Euler's gamma function. It is also considered the left fractional Riemann–Liouville derivative of order α , where $s-1 < \alpha < s$, with respect to coordinate x is $\overset{\alpha}{\partial}_x f(x) := \lim_{1x \rightarrow -\infty} {}_1x\overset{\alpha}{\partial}_x f(x)$.

Similarly, the right RL derivative is introduced by

$${}_x\overset{\alpha}{\partial}_{2x} f(x) := \frac{1}{\Gamma(s-\alpha)} \left(-\frac{\partial}{\partial x} \right)^s \int_x^{2x} (x'-x)^{s-\alpha-1} f(x') dx',$$

when the right fractional Liouville derivative is computed ${}_x\overset{\alpha}{\partial} f(x^k) := \lim_{2x \rightarrow \infty} {}_x\overset{\alpha}{\partial}_{2x} f(x)$.

Integro–differential constructions based only on RL derivatives seem to be very cumbersome for the purpose to elaborate (fractional) differential geometric models and has a number of properties which are very different from similar ones for the integer calculus. This problem is a consequence of the fact that the fractional RL derivative of a constant C is not zero but, for instance, ${}_1x\overset{\alpha}{\partial}_x C = C \frac{(x-1x)^{-\alpha}}{\Gamma(1-\alpha)}$. We shall use a different type of fractional derivative.

The fractional left, respectively, right Caputo derivatives are defined

$$\begin{aligned} {}_1x\overset{\alpha}{\underline{\partial}}_x f(x) &:= \frac{1}{\Gamma(s-\alpha)} \int_{1x}^x (x-x')^{s-\alpha-1} \left(\frac{\partial}{\partial x'} \right)^s f(x') dx'; \quad (1) \\ {}_x\overset{\alpha}{\underline{\partial}}_{2x} f(x) &:= \frac{1}{\Gamma(s-\alpha)} \int_x^{2x} (x'-x)^{s-\alpha-1} \left(-\frac{\partial}{\partial x'} \right)^s f(x') dx'. \end{aligned}$$

In the above formulas, we underline the partial derivative symbol, $\underline{\partial}$, in order to distinguish the Caputo differential operators from the RL ones with usual ∂ . For a constant C , for instance, ${}_1x\overset{\alpha}{\underline{\partial}}_x C = 0$.

The fractional absolute differential $\overset{\alpha}{d}$ is written in the form

$$\overset{\alpha}{d} := (dx^j)^\alpha \quad {}_0\overset{\alpha}{\underline{\partial}}_j, \quad \text{where} \quad \overset{\alpha}{dx}^j = (dx^j)^\alpha \frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)},$$

where we consider ${}_1x^i = 0$. For the "integer" calculus, we use as local coordinate co-bases/-frames the differentials $dx^j = (dx^j)^{\alpha=1}$. If $0 < \alpha < 1$, we have $dx = (dx)^{1-\alpha}(dx)^\alpha$. The "fractional" symbol $(dx^j)^\alpha$, related to ${}_1^\alpha dx^j$, is used instead of usual "integer" differentials (dual coordinate bases) dx^i for elaborating a co-vector/differential form calculus. We consider the exterior fractional differential ${}_1^\alpha d = \sum_{j=1}^n \Gamma(2-\alpha)(x^j)^{\alpha-1} {}_1^\alpha dx^j {}_0^\alpha \underline{\partial}_j$ and write the exact fractional differential 0-form as a fractional differential of the function ${}_1^\alpha d_x f(x) := (dx)^\alpha {}_1^\alpha \underline{\partial}_x f(x')$. The formula for the fractional exterior derivative is

$${}_1^\alpha d_x := (dx^i)^\alpha {}_1^\alpha \underline{\partial}_i. \quad (2)$$

For a fractional differential 1-form ${}^\alpha \tilde{\omega}$ with coefficients $\{\omega_i(x^k)\}$ we may write

$${}^\alpha \tilde{\omega} = (dx^i)^\alpha \omega_i(x^k) \quad (3)$$

when the exterior fractional derivatives of such a fractional 1-form ${}^\alpha \tilde{\omega}$ results a fractional 2-form, ${}_1^\alpha d_x ({}^\alpha \tilde{\omega}) = (dx^i)^\alpha \wedge (dx^j)^\alpha {}_1^\alpha \underline{\partial}_j \omega_i(x^k)$. The fractional exterior derivative (2) can be also written in the form

$${}_1^\alpha d_x := \frac{1}{\Gamma(\alpha+1)} {}_1^\alpha d_x (x^i - {}_1x^i)^\alpha {}_1^\alpha \underline{\partial}_i$$

when the fractional 1-form (3) is ${}^\alpha \tilde{\omega} = \frac{1}{\Gamma(\alpha+1)} {}_1^\alpha d_x (x^i - {}_1x^i)^\alpha F_i(x)$.

The above formulas introduce a well defined exterior calculus of fractional differential forms on flat spaces \mathbb{R}^n ; we can generalize the constructions for a real manifold M , $\dim M = n$. Any charts of a covering atlas of M can be endowed with a fractional derivative-integral structure of Caputo type as we explained above. Such a space \underline{M}^α (derived for a "prime" integer dimensional M of necessary smooth class) is called a fractional manifold. Similarly, the concept of tangent bundle can be generalized for fractional dimensions, using the Caputo fractional derivative. A tangent bundle TM over a manifold M is canonically defined by its local integer differential structure ∂_i . A fractional version is induced if instead of ∂_i we consider the left Caputo derivatives ${}_1^\alpha \underline{\partial}_i$ of type (1), for every local coordinate x^i on M . A fractional tangent bundle $\underline{T}M^\alpha$ for $\alpha \in (0, 1)$ (the symbol T is underlined in order to emphasize that we shall associate the approach to a fractional Caputo derivative). For simplicity, we shall write both for integer and fractional tangent bundles the local coordinates in the form $u^\beta = (x^j, y^j)$.

Any fractional frame basis $\underline{e}_\beta^\alpha = e^{\beta'}_\beta(u^\beta)\underline{\partial}_{\beta'}$ on $\underline{T}M$ is connected via a vielbein transform $e^{\beta'}_\beta(u^\beta)$ with a fractional local coordinate basis

$$\underline{\partial}_{\beta'} = \left(\underline{\partial}_{j'}^\alpha = {}_{1x^{j'}}^\alpha \underline{\partial}_{j'}, \underline{\partial}_{b'}^\alpha = {}_{1y^{b'}}^\alpha \underline{\partial}_{b'} \right), \quad (4)$$

for $j' = 1, 2, \dots, n$ and $b' = n+1, n+2, \dots, n+n$. The corresponding fractional co-bases are $\underline{e}^{\alpha\beta} = e^{\beta'}_\beta(u^\beta)\underline{du}^{\beta'}$, where the fractional local coordinate co-basis is written

$$\underline{du}^{\beta'} = \left((dx^{i'})^\alpha, (dy^{a'})^\alpha \right), \quad (5)$$

with h- and v-components, correspondingly, $(dx^{i'})^\alpha$ and $(dy^{a'})^\alpha$, being of type (3).

3 Fractional Lagrange–Finsler Geometry

A Lagrange space [45] $L^n = (M, L)$, of integer dimension n , is defined by a Lagrange fundamental function $L(x, y)$, i.e. a regular real function $L : TM \rightarrow \mathbb{R}$, for which the Hessian

$$Lg_{ij} = (1/2)\partial^2 L / \partial y^i \partial y^j$$

is not degenerate³.

We can consider that a Lagrange space L^n is a Finsler space F^n if and only if its fundamental function L is positive and two homogeneous with respect to variables y^i , i.e. $L = F^2$. For simplicity, we shall work with Lagrange spaces and their fractional generalizations, considering the Finsler ones to consist of a more particular, homogeneous, subclass.

Definition 3.1 A (target) fractional Lagrange space $\underline{L}^n = (\underline{M}, \underline{L})$ of fractional dimension $\alpha \in (0, 1)$, for a regular real function $\underline{L} : \underline{T}M \rightarrow \mathbb{R}$, when the fractional Hessian is

$${}_L \underline{g}_{ij}^\alpha = \frac{1}{4} \left(\underline{\partial}_i^\alpha \underline{\partial}_j^\alpha + \underline{\partial}_j^\alpha \underline{\partial}_i^\alpha \right) \underline{L}^\alpha \neq 0. \quad (6)$$

³the construction was used for elaborating a model of geometric mechanics following methods of Finsler geometry in a number of works on Lagrange–Finsler geometry and generalizations [39], see our developments for modern gravity and noncommutative spaces [46, 47]

In our further constructions, we shall use the coefficients ${}_L g^{ij}$ being inverse to ${}_L g_{ij}$ (6).⁴ Any \underline{L}^n can be associated to a prime "integer" Lagrange space L^n .

The concept of nonlinear connection (N-connection) on \underline{L}^n can be introduced similarly to that on nonholonomic fractional manifold [28, 29] considering the fractional tangent bundle $\underline{T}M$.

Definition 3.2 A N-connection $\overset{\alpha}{N}$ on $\underline{T}M$ is defined by a nonholonomic distribution (Whitney sum) with conventional h - and v -subspaces, $\underline{h} \underline{T}M$ and $\underline{v} \underline{T}M$, when

$$\overset{\alpha}{T} \overset{\alpha}{T} M = \underline{h} \overset{\alpha}{T} M \oplus \underline{v} \overset{\alpha}{T} M. \quad (7)$$

Locally, a fractional N-connection is defined by a set of coefficients, $\overset{\alpha}{N} = \{ {}^\alpha N_i^a \}$, when

$$\overset{\alpha}{N} = {}^\alpha N_i^a(u) (dx^i)^\alpha \otimes \underline{\partial}_a, \quad (8)$$

see local bases (4) and (5).

By an explicit construction, we prove

Proposition 3.1 There is a class of N-adapted fractional (co) frames linearly depending on ${}^\alpha N_i^a$,

$${}^\alpha \mathbf{e}_\beta = \left[{}^\alpha \mathbf{e}_j = \underline{\partial}_j - {}^\alpha N_j^a \underline{\partial}_a, {}^\alpha \mathbf{e}_b = \underline{\partial}_b \right], \quad (9)$$

$${}^\alpha \mathbf{e}^\beta = [{}^\alpha e^j = (dx^j)^\alpha, {}^\alpha \mathbf{e}^b = (dy^b)^\alpha + {}^\alpha N_k^b (dx^k)^\alpha]. \quad (10)$$

The above bases are nonholonomical (equivalently, non-integrable/ anholonomic) and characterized by the property that

$$[{}^\alpha \mathbf{e}_\alpha, {}^\alpha \mathbf{e}_\beta] = {}^\alpha \mathbf{e}_\alpha {}^\alpha \mathbf{e}_\beta - {}^\alpha \mathbf{e}_\beta {}^\alpha \mathbf{e}_\alpha = {}^\alpha W_{\alpha\beta}^\gamma {}^\alpha \mathbf{e}_\gamma,$$

where the nontrivial nonholonomy coefficients ${}^\alpha W_{\alpha\beta}^\gamma$ are computed ${}^\alpha W_{ib}^a = \underline{\partial}_b {}^\alpha N_i^a$ and ${}^\alpha W_{ij}^a = {}^\alpha \Omega_{ji}^a = {}^\alpha \mathbf{e}_i {}^\alpha N_j^a - {}^\alpha \mathbf{e}_j {}^\alpha N_i^a$; the values ${}^\alpha \Omega_{ji}^a$ define the coefficients of the N-connection curvature.

⁴We shall put a left label L to certain geometric objects if it is necessary to emphasize that they are induced by Lagrange generating function. Nevertheless, such labels will be omitted (in order to simplify the notations) if that will not result in ambiguities.

Let us consider values $y^k(\tau) = dx^k(\tau)/d\tau$, for $x(\tau)$ parametrizing smooth curves on a manifold M with $\tau \in [0, 1]$. The fractional analogs of such configurations are determined by changing $d/d\tau$ into the fractional Caputo derivative $\overset{\alpha}{\partial}_\tau = {}_{1\tau}\overset{\alpha}{\partial}_\tau$ when ${}^\alpha y^k(\tau) = \overset{\alpha}{\partial}_\tau x^k(\tau)$. For simplicity, we shall omit the label α for $y \in \overset{\alpha}{T}M$ if that will not result in ambiguities and/or we shall do not associate to it an explicit fractional derivative along a curve.

By straightforward computations, following the same schemes as in [39, 46, 47] but with fractional derivatives and integrals, we prove:

Theorem 3.1 *Any $\overset{\alpha}{L}$ defines the fundamental geometric objects determining canonically a nonholonomic fractional Riemann–Cartan geometry on $\overset{\alpha}{T}M$ being satisfied the properties:*

1. *The fractional Euler–Lagrange equations*

$$\overset{\alpha}{\partial}_\tau \left({}_{1y^i}\overset{\alpha}{\partial}_i \overset{\alpha}{L} \right) - {}_{1x^i}\overset{\alpha}{\partial}_i \overset{\alpha}{L} = 0$$

are equivalent to the fractional "nonlinear geodesic" (equivalently, semi-spray) equations

$$\left(\overset{\alpha}{\partial}_\tau \right)^2 x^k + 2G^k(x, {}^\alpha y) = 0,$$

where

$$G^k = \frac{1}{4} {}_L g^{kj} \left[y^j {}_{1y^j}\overset{\alpha}{\partial}_j \left({}_{1x^i}\overset{\alpha}{\partial}_i \overset{\alpha}{L} \right) - {}_{1x^i}\overset{\alpha}{\partial}_i \overset{\alpha}{L} \right]$$

defines the canonical N-connection

$${}^\alpha_L N_j^a = {}_{1y^j}\overset{\alpha}{\partial}_j G^k(x, {}^\alpha y). \quad (11)$$

2. *There is a canonical (Sasaki type) metric structure,*

$${}^\alpha_L \mathbf{g} = {}^\alpha_L g_{kj}(x, y) {}^\alpha e^k \otimes {}^\alpha e^j + {}^\alpha_L g_{cb}(x, y) {}^\alpha_L \mathbf{e}^c \otimes {}^\alpha_L \mathbf{e}^b, \quad (12)$$

*where the preferred frame structure (defined linearly by ${}^\alpha_L N_j^a$) is ${}^\alpha_L \mathbf{e}_\nu = ({}^\alpha_L \mathbf{e}_i, e_a)$.*⁵

⁵ A (fractional) general metric structure $\overset{\alpha}{\mathbf{g}} = \{ {}^\alpha g_{\alpha\beta} \}$ is defined on $\overset{\alpha}{T}M$ by a symmetric second rank tensor $\overset{\alpha}{\mathbf{g}} = {}^\alpha g_{\gamma\beta}(u)(du^\gamma)^\alpha \otimes (du^\beta)^\alpha$. For N-adapted constructions, see

3. There is a canonical metrical distinguished connection

$${}_c^{\alpha}\mathbf{D} = (h {}_c^{\alpha}D, v {}_c^{\alpha}D) = \{ {}_c^{\alpha}\mathbf{\Gamma}_{\alpha\beta}^{\gamma} = ({}^{\alpha}\widehat{L}_{jk}^i, {}^{\alpha}\widehat{C}_{jc}^i) \},$$

(in brief, *d*-connection), which is a linear connection preserving under parallelism the splitting (7) and metric compatible, i.e. ${}_c^{\alpha}\mathbf{D} \left({}_L^{\alpha}\mathbf{g} \right) = 0$, when

$$\begin{aligned} {}_c^{\alpha}\mathbf{\Gamma}_j^i &= {}_c^{\alpha}\mathbf{\Gamma}_{j\gamma}^i {}_L^{\alpha}\mathbf{e}^{\gamma} = \widehat{L}_{jk}^i e^k + \widehat{C}_{jc}^i {}_L^{\alpha}\mathbf{e}^c, \\ \text{for } \widehat{L}_{jk}^i &= \widehat{L}_{bk}^a, \widehat{C}_{jc}^i = \widehat{C}_{bc}^a \text{ in } {}_c^{\alpha}\mathbf{\Gamma}_b^a = {}_c^{\alpha}\mathbf{\Gamma}_{b\gamma}^a {}_L^{\alpha}\mathbf{e}^{\gamma} = \widehat{L}_{bk}^a e^k + \widehat{C}_{bc}^a {}_L^{\alpha}\mathbf{e}^c, \\ \text{and} \end{aligned}$$

$$\begin{aligned} {}^{\alpha}\widehat{L}_{jk}^i &= \frac{1}{2} {}_L^{\alpha}g^{ir} ({}_L^{\alpha}\mathbf{e}_k {}_L^{\alpha}g_{jr} + {}_L^{\alpha}\mathbf{e}_j {}_L^{\alpha}g_{kr} - {}_L^{\alpha}\mathbf{e}_r {}_L^{\alpha}g_{jk}), \\ {}^{\alpha}\widehat{C}_{bc}^a &= \frac{1}{2} {}_L^{\alpha}g^{ad} ({}_c^{\alpha}e_c {}_L^{\alpha}g_{bd} + {}_c^{\alpha}e_c {}_L^{\alpha}g_{cd} - {}_c^{\alpha}e_d {}_L^{\alpha}g_{bc}) \end{aligned} \quad (13)$$

are just the generalized Christoffel indices.⁶

Finally, in this section, we note that:

Remark 3.1 We note that ${}_c^{\alpha}\mathbf{D}$ is with nonholonomically induced torsion structure defined by 2-forms

$$\begin{aligned} {}_L^{\alpha}\mathcal{T}^i &= \widehat{C}_{jc}^i {}^{\alpha}e^j \wedge {}_L^{\alpha}\mathbf{e}^c, \\ {}_L^{\alpha}\mathcal{T}^a &= -\frac{1}{2} {}_L\Omega_{ij}^a {}^{\alpha}e^i \wedge {}^{\alpha}e^j + \left({}^{\alpha}e_b {}_L N_i^a - {}^{\alpha}\widehat{L}_{bi}^a \right) {}^{\alpha}e^i \wedge {}_L^{\alpha}\mathbf{e}^b \end{aligned} \quad (14)$$

details in [28, 29], it is important to use the property that any fractional metric ${}^{\alpha}\mathbf{g}$ can be represented equivalently as a distinguished metric (*d*-metric), ${}^{\alpha}\mathbf{g} = [{}^{\alpha}g_{kj}, {}^{\alpha}g_{cb}]$, when

$$\begin{aligned} {}^{\alpha}\mathbf{g} &= {}^{\alpha}g_{kj}(x, y) {}^{\alpha}e^k \otimes {}^{\alpha}e^j + {}^{\alpha}g_{cb}(x, y) {}^{\alpha}\mathbf{e}^c \otimes {}^{\alpha}\mathbf{e}^b \\ &= \eta_{k'j'} {}^{\alpha}e^{k'} \otimes {}^{\alpha}e^{j'} + \eta_{c'b'} {}^{\alpha}\mathbf{e}^{c'} \otimes {}^{\alpha}\mathbf{e}^{b'}, \end{aligned}$$

where matrices $\eta_{k'j'} = \text{diag}[\pm 1, \pm 1, \dots, \pm 1]$ and $\eta_{a'b'} = \text{diag}[\pm 1, \pm 1, \dots, \pm 1]$ (reflecting signature of "prime" space TM) are obtained by frame transforms $\eta_{k'j'} = e_{k'}^k e_{j'}^j {}^{\alpha}g_{kj}$ and $\eta_{a'b'} = e_{a'}^a e_{b'}^b {}^{\alpha}g_{ab}$. For fractional computations, it is convenient to work with constants $\eta_{k'j'}$ and $\eta_{a'b'}$ because the Caputo derivatives of constants are zero. This allows us to keep the same tensor rules as for the integer dimensions even the rules for taking local derivatives became more sophisticated because of N-coefficients ${}^{\alpha}N_i^a(u)$ and additional vierbein transforms $e_{k'}^k(u)$ and $e_{a'}^a(u)$. Such coefficients mix fractional derivatives $\widehat{\partial}_a$ computed as a local integration (1). If we work with RL fractional derivatives, the computation become very sophisticated with nonlinear mixing of integration, partial derivatives etc. For some purposes, just RL may be very important. Our ideas, for such cases, is to derive a geometric model with the Caputo fractional derivatives, and after that we can re-define the nonholonomic frames/distributions in a form to extract certain constructions with the RL derivative.

⁶for integer dimensions, we contract "horizontal" and "vertical" indices following the rule: $i = 1$ is $a = n + 1$; $i = 2$ is $a = n + 2$; ... $i = n$ is $a = n + n$ "

computed from the fractional version of Cartan's structure equations

$$\begin{aligned} d \, {}^\alpha e^i - {}^\alpha e^k \wedge {}^\alpha \Gamma^i_k &= - {}^\alpha \mathcal{T}^i, \\ d \, {}^\alpha \mathbf{e}^a - {}^\alpha \mathbf{e}^b \wedge {}^\alpha \Gamma^a_b &= - {}^\alpha \mathcal{T}^a, \\ d \, {}^\alpha \Gamma^i_j - {}^\alpha \Gamma^k_j \wedge {}^\alpha \Gamma^i_k &= - {}^\alpha \mathcal{R}^i_j \end{aligned} \quad (15)$$

in which the curvature 2-form is denoted ${}^\alpha \mathcal{R}^i_j$.

In general, for any d-connection on ${}^\alpha \underline{T}M$, we can compute respectively the N-adapted coefficients of torsion ${}^\alpha \mathcal{T}^\tau = \{ {}^\alpha \Gamma^\tau_{\beta\gamma} \}$ and curvature ${}^\alpha \mathcal{R}^\tau_\beta = \{ {}^\alpha \mathbf{R}^\tau_{\beta\gamma\delta} \}$ as it is explained for general fractional nonholonomic manifolds in [28, 29].

4 Almost Kähler Models of Fractional Lagrange Spaces

The goals of this section is to prove that the canonical N-connection ${}^\alpha N$ (11) induces an almost Kähler structure defined canonically by a fractional regular ${}^\alpha \tilde{L}(x, {}^\alpha y)$.

Definition 4.1 *A fractional nonholonomic almost complex structure is defined as a linear operator ${}^\alpha \mathbf{J}$ acting on the vectors on ${}^\alpha \underline{T}M$ following formulas*

$${}^\alpha \mathbf{J}({}^\alpha \mathbf{e}_i) = - {}^\alpha e_i \text{ and } {}^\alpha \mathbf{J}({}^\alpha e_i) = {}^\alpha \mathbf{e}_i,$$

where the superposition ${}^\alpha \mathbf{J} \circ {}^\alpha \mathbf{J} = -\mathbf{I}$, for \mathbf{I} being the unity matrix.

The fractional operator ${}^\alpha \mathbf{J}$ reduces to a complex structure \mathbb{J} if and only if the distribution (7) is integrable and the dimensions are taken to be some integer ones.

Lemma 4.1 –Definition. *A fractional ${}^\alpha \tilde{L}$ induces a canonical 1-form ${}^\alpha \omega = \frac{1}{2} \left({}_{1y^i} {}^\alpha \underline{\partial}_i {}^\alpha \tilde{L} \right) {}^\alpha e^i$ and a metric ${}^\alpha \mathbf{g}$ (12) induces a canonical 2-form*

$${}^\alpha \theta = {}^\alpha \mathbf{g}_{ij}(x, {}^\alpha y) {}^\alpha \mathbf{e}^i \wedge {}^\alpha e^j. \quad (16)$$

associated to ${}^\alpha \mathbf{J}$ following formulas ${}^\alpha \theta(\mathbf{X}, \mathbf{Y}) \doteq {}^\alpha \mathbf{g} \left({}^\alpha \mathbf{J}\mathbf{X}, \mathbf{Y} \right)$ for any vectors \mathbf{X} and \mathbf{Y} on ${}^\alpha \underline{T}M$.

Using a fractional N-adapted form calculus,

$$\begin{aligned} d \, {}^\alpha_L \theta &= \frac{1}{6} \sum_{(ijk)} {}^L \tilde{g}_{is} {}^\alpha_L \Omega_{jk}^s {}^\alpha e^i \wedge {}^\alpha e^j \wedge {}^\alpha e^k \\ &+ \frac{1}{2} \left({}^L \tilde{g}_{ij||k} - {}^L \tilde{g}_{ik||j} \right) {}^\alpha_L \mathbf{e}^i \wedge {}^\alpha e^j \wedge {}^\alpha e^k \\ &+ \frac{1}{2} \left({}^\alpha e_k ({}^L \tilde{g}_{ij}) - {}^\alpha e_i ({}^L \tilde{g}_{kj}) \right) {}^\alpha_L \mathbf{e}^k \wedge {}^\alpha_L \mathbf{e}^i \wedge {}^\alpha e^j, \end{aligned}$$

where (ijk) means symmetrization of indices and

$${}^L \tilde{g}_{ij||k} := {}^\alpha_L \mathbf{e}_k \, {}^L \tilde{g}_{ij} - {}^L B_{ik}^s \, {}^L \tilde{g}_{sj} - {}^L B_{jk}^s \, {}^L \tilde{g}_{is},$$

for ${}^L B_{ik}^s = {}^\alpha e_i \, {}^\alpha N_k^s$, we prove the results:

Proposition 4.1 1. A regular L defines on TM an almost Hermitian (symplectic) structure ${}^\alpha_L \theta$ for which $d \, {}^\alpha_L \omega = {}^\alpha_L \theta$;

2. The canonical N-connection ${}^\alpha N_j^a$ (11) and its curvature have the properties

$$\begin{aligned} \sum_{ijk} {}^L \tilde{g}_{l(i} {}^\alpha_L \Omega_{jk)}^l &= 0, \quad {}^L \tilde{g}_{ij||k} - {}^L \tilde{g}_{ik||j} = 0, \\ {}^\alpha e_k ({}^L \tilde{g}_{ij}) - {}^\alpha e_j ({}^L \tilde{g}_{ik}) &= 0. \end{aligned}$$

Conclusion 4.1 The above properties state an almost Hermitian model of fractional Lagrange space $\underline{\tilde{L}}^n = (\underline{\tilde{M}}, \underline{\tilde{L}})$ as a fractional almost Kähler manifold with $d \, {}^\alpha_L \theta = 0$. The triad ${}^\alpha \mathbb{K}^{2n} = (\underline{\tilde{T}M}, \, {}^\alpha_L \tilde{\mathbf{g}}, \mathbf{J})$, where $\underline{\tilde{T}M} := \underline{\tilde{T}M} \setminus \{0\}$ for $\{0\}$ denoting the null-sections under $\underline{\tilde{M}}$, defines a fractional, and non-holonomic, almost Kähler space.

Our next purpose is to construct a canonical (i.e. uniquely determined by \tilde{L}) almost Kähler distinguished connection (d-connection) ${}^\theta \tilde{\mathbf{D}}$ being compatible both with the almost Kähler $\left({}^\alpha_L \theta, \mathbf{J} \right)$ and N-connection structures ${}^\alpha_L \mathbf{N}$ when

$${}^\theta \tilde{\mathbf{D}}_{\mathbf{X}} \, {}^\alpha_L \tilde{\mathbf{g}} = \mathbf{0} \text{ and } {}^\theta \tilde{\mathbf{D}}_{\mathbf{X}} \mathbf{J} = \mathbf{0}, \quad (17)$$

for any vector $\mathbf{X} = X^i \, {}^\alpha_L \mathbf{e}_i + X^a \, {}^\alpha e_a$. By a straightforward computation, we prove (see similar "integer" details in [39, 46, 31]):

Theorem 4.1 (Main Result) *The fractional canonical metrical d -connection ${}^\alpha\mathbf{D}$, possessing N -adapted coefficients $(\hat{L}^a_{bk}, \hat{C}^a_{bc})$ (13), defines a (unique) canonical fractional almost Kähler d -connection ${}^\theta\mathbf{D} = {}^\alpha\mathbf{D}$ satisfying the conditions (17).*

Finally, we note that:

Remark 4.1 *There are two important particular cases:*

1. If ${}^\alpha\hat{L} = ({}^\alpha\hat{F})^2$, for a Finsler space of an integer dimension, we get a canonical almost Kähler model of Finsler space [40], when ${}^\theta\mathbf{D} = {}^\alpha\mathbf{D}$ transforms in the so-called Cartan–Finsler connection [38] (for integer dimensional Lagrange spaces, a similar result was proven in [41], see details in [39]).
2. We get a Kählerian model of a Lagrange, or Finsler, space if the respective almost complex structure \mathbf{J} is integrable. This property holds also for respective spaces of fractional dimension.

Finally, we emphasize that the geometric background for fractional Lagrange–Finsler spaces provided in this section has been used in our partner work [30] for deformation quantization of such theories. This is an explicit proof that fractional physical models can be quantized in general form, that it is possible to elaborate new types/models/theories of quantum fractional calculus and geometries stating a new unified formalism to fractional classical and quantum geometries and physics and propose various applications in mechanics and engineering.

5 Conclusions

The fractional Caputo derivative allows us to formulate a self-consistent nonholonomic geometric approach to fractional calculus [28, 29] when various important mathematical and physical problems for spaces and processes of non-integer dimension can be investigated using various methods originally elaborated in modern Lagrange–Hamilton–Finsler geometry and generalizations. In this paper, we provided a geometrization of regular fractional Lagrange mechanics in a form similar to that in Finsler geometry but, in our case, extended to non-integer dimensions. This way, applications of

fractional calculus become a direction of non-Riemannian geometry with nonholonomic distributions and generalized connections.

Nonholonomic fractional mechanical interactions are described by a sophisticated variational calculus and derived cumbersome systems of nonlinear equations and less defined types of symmetries. To quantize such mechanical systems is a very difficult problem related to a number of unsolved fundamental problems of nonlinear functional analysis. Nevertheless, for certain approaches based on fractional Caputo derivative, or those admitting nonholonomic deformations/ transforms to such fractional configurations, it is possible to provide a geometrization of fractional Lagrange mechanics. In this case, the problem of quantization can be approached following powerful methods of geometric and/or deformation quantization.

Having a similarity between fractional Lagrange geometry and certain generalized almost Kähler-Finsler models, for which the Fedosov quantization was performed in our previous works [31, 32, 33], our main goal was to prove the Theorem 4.1. As a result, we obtained a suitable geometric background for Fedosov quantization of fractional Lagrange-Finsler spaces which will be performed in our partner work [30].

Finally, we conclude that the examples of fractional almost Kähler spaces considered in this works can be re-defined for another types of nonholonomic distributions and applied, for instance, as a geometric scheme for deformation/ A-brane quantization of fractional Hamilton and Einstein spaces and various generalizations, similarly to our former constructions performed for integer dimensions in Refs. [34, 42, 43]. Such developments consist a purpose of our further research.

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